Cramér-Rao Bound for Localization with A Priori Knowledge on Biased Range Measurements

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Abstract

This paper derives a general expression for the Cramér-Rao bound (CRB) of wireless localization algorithms using range measurements subject to bias corruption. Specifically, the a priori knowledge about which range measurements are biased, and the probability density functions (PDF) of the biases are assumed to be available. For each range measurement, the error due to estimating the time-of-arrival of the detected signal is modeled as a Gaussian distributed random variable with zero mean and known variance. In general, the derived CRB expression can be evaluated numerically. An approximate CRB expression is also derived when the bias PDF is very informative. Using these CRB expressions, we study the impact of the bias distribution on the mean square error (MSE) bound corresponding to the CRB. The analysis is corroborated by numerical experiments.

Index Terms

Cramér-Rao bound, localization, biased range measurements, wireless networks.

I. Introduction

Wireless localization systems have been attracting intensive research interest in both academia and industry lately. For indoor or dense urban environments, Global Positioning Systems do not function well for geolocation purposes. Instead, a few beacons at fixed and known positions are exploited for geolocation using measurements related to signal power decay, bearing, difference of range, and range between a

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target and the beacons [1]. Thanks to the superior multipath resolution and penetration capability of ultra-wideband (UWB), a range measurement can be obtained by UWB pulsed signals with high accuracy [2]. Therefore, range based localization has become a promising option for short-range wireless networks.

Much work on ranging algorithms based on time of arrival (TOA) measurements has been reported [2], [3]. Specifically, they are usually based on detecting the first arriving signal from a transmitter and estimating its time of arrival (TOA) at a receiver. Note that the clocks of the transmitter and the receiver need to synchronized, and the transmission time at the transmitter should be sent to the receiver for producing the range measurements. In this paper, we consider the geolocation algorithms using range measurements generated between the target and multiple beacons in the aforementioned way. It is important to note that the considered algorithms are suitable for cooperative (blue-force) geolocation applications, in that they require the cooperation of the target and the beacons in order to synchronize their clocks for generating range measurements.

In general, a range measurement error can be modeled as the sum of two terms. The first one is due to the error of estimating the TOA of the detected signal, while the second one corresponds to the difference between the path length traveled by the detected signal and the transmitter-receiver distance. For instance, the second term is nonzero when the detected signal does not come from the line of sight propagation path. The first term is usually modeled as a zero mean random variable with a Gaussian distribution in the literature [4]–[6]. As a consequence, the second term is equal to the bias of the range measurement.

It is well known that the Cramér-Rao bound (CRB) sets a fundamental lower limit to the covariance of any unbiased estimator for a vector parameter (please refer to pages 63 to 72 in [7]). Specifically, the mean square error (MSE) of any unbiased location estimator is lower bounded by the corresponding CRB's trace, referred to as the MSE bound hereafter. The CRB and the MSE bound are used extensively to evaluate particular localization algorithms, as well as guide the geolocation system design fulfilling certain accuracy requirements [4], [8], [9]. When some of the range measurements are biased, the CRB has been derived in [10] by regarding each bias as a deterministic nuisance parameter, and it was shown in [5] that using those bias-corrupted measurements does not improve the CRB.

Recently, much work has been reported on modeling the bias as a random variable with a particular probability density function (PDF). Specifically, this PDF is obtained empirically by extensive experiments [11], [12], or theoretically by scattering models [13]. In case the PDF of the bias is known a priori, a Cramér-Rao-like bound has been derived in [14] for the joint estimation of the deterministic target location and the random bias. This bound is referred to as the hybrid CRB in [15], [16], since the parameters to be estimated consist of both deterministic and random parameters. It was shown in [5] that using the

bias-corrupted measurements improves the MSE bound associated with the hybrid CRB.

Specifically, the hybrid CRB was derived from the joint PDF of the distance measurements and the random bias. Compared with the hybrid CRB, the CRB derived from the marginal PDF of range measurements is much tighter and can be asymptotically attained by a maximum likelihood (ML) estimator [7], [16]. When the bias PDF is approximated by a piecewise constant function, an expression that can be numerically evaluated was derived for the CRB in [6]. Numerical results showed that for uniformly distributed bias, the presence of the bias degrades the MSE bound and using the bias-corrupted measurements improves the MSE bound.

Compared with the above existing work, this paper contains the following contributions:

- We derive a general expression that can be numerically evaluated for the CRB. When the prior bias PDF is very informative, an approximate CRB expression is derived.
- Based on these expressions, the impact of the bias distribution on the MSE bound corresponding to the trace of the CRB is studied analytically.

The rest of this paper is organized as follows. Section II describes a typical 2D localization system and related models for range measurements. In Section III, a general CRB expression is derived for this system. After that, an approximate CRB expression is derived in Section IV when the prior bias PDF is very informative. Based on these expressions, the impact of the bias distribution on the MSE bound is studied analytically in Section V. We will show some numerical results in Section VI, and complete this paper by some conclusions in Section VII.

Notations: Upper (lower) boldface letters denote matrices (column vectors), and $[\cdot]^T$ represents the transpose operator. $\nabla_{\mathbf{x}}(f(\mathbf{x}))$ stands for a column vector which is the gradient of the function $f(\mathbf{x})$ with respect to $\mathbf{x}$, and $\nabla^2_{\mathbf{x}}(f(\mathbf{x}))$ represents a matrix which is the Hessian of the function $f(\mathbf{x})$ with respect to $\mathbf{x}$.

II. SYSTEM SETUP AND RANGING MODELS

We consider a typical 2D geolocation system equipped with M beacons $(M \geq 3)$. In the following sections, although only the CRB for this 2D system is derived, the CRB for a 3D system can be derived in the same way and will be given as well. The coordinate of beacon m is $\mathbf{p}_m = [x_m, y_m]^T (m = 1, \cdots, M)$, and a target is located at $\mathbf{u} = [x_{\mathrm{u}}, y_{\mathrm{u}}]^T$.

Suppose independent range measurements have been produced between a target and beacons, and the one between beacon m and the target is denoted by r_m. Without loss of generality, we assume the first L range measurements are biased, while the other measurements are all unbiased. The measurement error

related to r_m is modeled as:

$$e_m = r_m - d_m = \begin{cases} \varepsilon_m + b_m & m = 1, \cdots, L \\ \varepsilon_m & m = L+1, \cdots, M, \end{cases} \tag{1}$$

where d_m represents the distance between the target and beacon m, and ε_m is due to the TOA estimation error of the detected signal, and b_m corresponds to the difference of the path length traveled by the detected signal and the distance from beacon m to the target. We assume b_m has the a priori known PDF $p(b_m)$ with its mean and variance denoted by $\overline{b}_m$ and κ_m^2, respectively. In addition, we assume b_m is independent of ε_m, and ε_m is Gaussian distributed with zero mean. Besides, we denote its variance as σ_m^2 and assume it is known a priori. The motivation behind this assumption is twofold. One is that it simplifies the mathematical derivation. The other is that the impact of the bias distribution on the localization CRB can be studied when the TOA estimation has a guaranteed accuracy at a prescribed level.

We stack all range measurements into the column vector $\mathbf{r} = [r_1, \cdots, r_M]^T$. The log-likelihood function of $\mathbf{u}$ is denoted as $\Lambda = \ln(p(\mathbf{r}; \mathbf{u}))$ where $p(\mathbf{r}; \mathbf{u})$ is the PDF of $\mathbf{r}$. Since all range measurements are independent of each other, $\Lambda = \sum_{m=1}^{M} \Lambda_m$, where $\Lambda_m = \ln(p(r_m; \mathbf{u}))$ is the log-likelihood function of r_m given $\mathbf{u}$, and $p(r_m; \mathbf{u})$ is the PDF of r_m.

For an unbiased range measurement r_m $(m = L+1, \cdots, M)$, $p(r_m; \mathbf{u})$ can be formulated as:

$$p(r_m; \mathbf{u}) = \frac{1}{\sqrt{2\pi\sigma_m^2}} \exp\left[-\frac{(r_m - d_m)^2}{2\sigma_m^2}\right] \tag{2}$$

For a biased range measurement r_m $(m = 1, \cdots, L)$, we define $\Gamma_m = \ln(p(r_m | b_m; \mathbf{u}))$ as the joint log-likelihood function of r_m given $\mathbf{u}$ and b_m, where $p(r_m | b_m; \mathbf{u})$ is the PDF of r_m conditional on b_m and $\mathbf{u}$. $p(r_m | b_m; \mathbf{u})$ and $p(r_m; \mathbf{u})$ can be expressed respectively as:

$$p(r_m | b_m; \mathbf{u}) = \frac{1}{\sqrt{2\pi\sigma_m^2}} \exp\left[-\frac{(r_m - d_m - b_m)^2}{2\sigma_m^2}\right] \tag{3}$$

$$p(r_m; \mathbf{u}) = \int_{-\infty}^{+\infty} p(r_m | b_m; \mathbf{u}) p(b_m) \mathrm{d}b_m \tag{4}$$

III. Derivations of a general CRB expression

The CRB is the inverse of the Fisher information matrix (FIM) $\mathbf{F}$, which is expressed by:

$$\mathbf{F} = \mathbb{E}_{\mathbf{r};\mathbf{u}}\left[(\nabla_{\mathbf{u}}\Lambda)(\nabla_{\mathbf{u}}\Lambda)^T\right] \tag{5}$$

where $\mathbb{E}_{\mathbf{r};\mathbf{u}}[\cdot]$ denotes the expectation operator with respect to $p(\mathbf{r}; \mathbf{u})$. Since all distance measurements are independent of each other, $\mathbf{F} = \sum_{m=1}^{M} \mathbf{F}_m$ where $\mathbf{F}_m$ is the FIM related to r_m [7]:

$$\mathbf{F}_m = \mathbb{E}_{r_m;\mathbf{u}}\left[(\nabla_{\mathbf{u}}\Lambda_m)(\nabla_{\mathbf{u}}\Lambda_m)^T\right] \tag{6}$$

where $\mathbb{E}_{r_m;\mathbf{u}}[\cdot]$ denotes the expectation operator with respect to $p(r_m;\mathbf{u})$.

When $m = L+1, \cdots, M$, we can show that:

$$\begin{aligned}\mathbf{F}_m &= \mathbb{E}_{r_m;\mathbf{u}}\left[\frac{(r_m - d_m)^2}{\sigma_m^4}\boldsymbol{\phi}_m\boldsymbol{\phi}_m^T\right] \\ &= \sigma_m^{-2}\boldsymbol{\phi}_m\boldsymbol{\phi}_m^T, \end{aligned} \tag{7}$$

where $\boldsymbol{\phi}_m = \frac{\mathbf{u}-\mathbf{p}_m}{d_m}$ represents the orientation of the target relative to beacon m.

When $m = 1, \cdots, L$, we can first prove that:

$$\begin{aligned}\nabla_{\mathbf{u}}\Lambda_m &= \frac{1}{p(r_m;\mathbf{u})}\int_{-\infty}^{+\infty}\left[\nabla_{\mathbf{u}}p(r_m|b_m;\mathbf{u})\right]p(b_m)\mathrm{d}b_m \\ &= \int_{-\infty}^{+\infty}\frac{p(r_m|b_m;\mathbf{u})p(b_m)}{p(r_m;\mathbf{u})}\left[\nabla_{\mathbf{u}}\Gamma_m\right]\mathrm{d}b_m \\ &= \mathbb{E}_{b_m|r_m;\mathbf{u}}\left[\nabla_{\mathbf{u}}\Gamma_m\right] \\ &= \sigma_m^{-2}a_m\boldsymbol{\phi}_m \end{aligned} \tag{8}$$

where $a_m = \mathbb{E}_{b_m|r_m;\mathbf{u}}[r_m - d_m - b_m]$ and $\mathbb{E}_{b_m|r_m;\mathbf{u}}[\cdot]$ denotes the expectation operator with respect to the posterior PDF $p(b_m|r_m;\mathbf{u})$. This means that a_m actually stands for the posterior mean of ε_m conditional on r_m. Inserting (8) into (6), we can compute $\mathbf{F}_m$ by:

$$\mathbf{F}_m = \lambda_m\boldsymbol{\phi}_m\boldsymbol{\phi}_m^T \tag{9}$$

where $\lambda_m = \sigma_m^{-4}\cdot\mathbb{E}_{r_m;\mathbf{u}}\left[a_m^2\right]$.

Summing $\mathbf{F}_1, \cdots, \mathbf{F}_M$ and taking the inverse, we can express the CRB as:

$$\mathbf{C} = \left[\sum_{m=1}^{L}\lambda_m\boldsymbol{\phi}_m\boldsymbol{\phi}_m^T + \sum_{m=L+1}^{M}\sigma_m^{-2}\boldsymbol{\phi}_m\boldsymbol{\phi}_m^T\right]^{-1} \tag{10}$$

The CRB for a 3-D localization system can be derived in the same way as given above. Most interestingly, we can show after simple mathematical arrangement that the CRB for the 3-D localization system can still be evaluated by (10), when $\mathbf{u}$ and $\mathbf{p}_m$ are substituted with the 3-D coordinates of the target and beacon m, respectively.

It is important to note that (10) is a general expression for the CRB, when each biased measurement r_m is Gaussian distributed conditional on b_m. However, we have proven that $\nabla_{\mathbf{u}}\Lambda_m = \mathbb{E}_{b_m|r_m;\mathbf{u}}[\nabla_{\mathbf{u}}\Gamma_m]$ holds in general for any particular form of $p(r_m|b_m;\mathbf{u})$. This means that a closed-form CRB expression can be derived in a similar way when $p(r_m|b_m;\mathbf{u})$ takes other specific form.

To illustrate the generality of (10), we show that the CRB can be derived by (10) when b_m is deterministic. Note that this CRB will be used when illustrating the effectiveness of the approximate CRB derived in

Section IV. In this case, $p(b_m|r_m;\mathbf{u}) = \delta(b_m - \overline{b}_m)$, $a_m = r_m - d_m - \overline{b}_m$, $p(r_m;\mathbf{u}) = p(r_m|b_m = \overline{b}_m;\mathbf{u})$, and therefore $\lambda_m = \sigma_m^{-2}$. This means that the contribution of r_m to the CRB is the same as if this range measurement were not bias corrupted. This is true since when b_m is deterministic and known a priori, its value can be subtracted from r_m, which means that r_m is in effect not subject to the bias corruption.

In general, an analytical expression for λ_m does not exist, in such a case λ_m can be evaluated numerically, with $\mathbf{C}$ following from equation (10). This provides a way to study the effects of the geometric configuration, the TOA estimation quality, as well as the bias distribution on the CRB. To this end, it can easily be shown that

$$\lambda_m = \sigma_m^{-4} \int_{-\infty}^{+\infty} a_m^2 p(r_m;\mathbf{u}) \mathrm{d}r_m \tag{11}$$

and

$$a_m = \int_{-\infty}^{+\infty} (r_m - d_m - b_m) \frac{p(r_m|b_m;\mathbf{u})p(b_m)}{p(r_m;\mathbf{u})} \mathrm{d}b_m, \tag{12}$$

respectively. For every value of r_m, $p(r_m;\mathbf{u})$ and a_m can be first evaluated by computing numerically the integrals in (4) and (12) with respect to b_m [1], respectively. Then, λ_m can be evaluated by computing numerically the integral with respect to r_m in (11).

Note that the CRB expression in (10), as those presented in [4]–[6], is derived under the assumption that we have the a priori knowledge about which range measurements are biased. In almost all practical scenarios, this a priori knowledge may not be available [18]–[20]. In this case, the CRB in (10) is an optimistic lower bound, since it assumes the implicit knowledge which is not really available for the range measurements. We will illustrate the optimism of (10) when the a priori knowledge about which range measurements are biased is not available by numerical experiments in Section VI. Relaxing the assumption of knowing which measurements are biased, and generating an information reduction factor that scales the CRB when the probability of each measurement getting biased is known a priori, in a similar way as shown in [18]–[20], would be a valuable piece of future work.

We should also note that the derived CRB is for cooperative geolocation applications, since the synchronization between the target and the beacons is required as mentioned in the introduction part. For geolocating evasive targets, range measurements would only be available with active sensing, in which case azimuth measurements, etc., would be available as well. It is another interesting piece of future work to derive the CRB for localization using azimuth measurements, etc., in addition to range measurements.

[1]The detailed procedure of numerical integration can be found in pages $198 - 200$ in [17].

IV. Derivation of an approximate CRB expression when $\kappa_m \ll \sigma_m$

We now derive an approximate CRB expression when b_m has a small variance compared to σ_m (i.e. $\kappa_m \ll \sigma_m$). First of all, we express $\mathbf{F}_m (m = 1, \cdots, L)$ alternatively as [7]:

$$\mathbf{F}_m = -\mathbb{E}_{r_m;\mathbf{u}} \left[\nabla^2_{\mathbf{u}} \Lambda_m\right] \tag{13}$$

Next, we approximate $p(r_m|b_m;\mathbf{u})$ by the first-order Taylor-series expansion as follows:

$$p(r_m|b_m;\mathbf{u}) \approx f_m + f'_m \cdot (b_m - \overline{b}_m) \tag{14}$$

where f_m and f'_m are expressed respectively as:

$$f_m = p(r_m|b_m = \overline{b}_m;\mathbf{u}) \tag{15}$$

$$f'_m = \frac{\partial p(r_m|b_m;\mathbf{u})}{\partial b_m}\Big|_{b_m=\overline{b}_m} \tag{16}$$

Based on (4) and (14), we can show that $p(r_m;\mathbf{u}) \approx f_m$. In addition, we make the following approximations:

$$\begin{aligned}\frac{p(r_m|b_m;\mathbf{u})}{p(r_m;\mathbf{u})} &\approx \frac{f_m + f'_m \cdot (b_m - \overline{b}_m)}{f_m} \\ &= 1 + \frac{\partial \Gamma_m}{\partial b_m}\Big|_{b_m=\overline{b}_m} \cdot (b_m - \overline{b}_m) \\ &= 1 + \frac{r_m - d_m - \overline{b}_m}{\sigma_m^2}(b_m - \overline{b}_m)\end{aligned} \tag{17}$$

Inserting (17) into (12), we can approximate a_m as follows:

$$a_m \approx \left(1 - \frac{\kappa_m^2}{\sigma_m^2}\right)(r_m - d_m - \overline{b}_m) \tag{18}$$

Based on (8) and (18), $\nabla^2_{\mathbf{u}}\Lambda_m$ can be approximated by:

$$\begin{aligned}\nabla^2_{\mathbf{u}}\Lambda_m &= \sigma_m^{-2}\left[(\nabla_{\mathbf{u}} a_m)\boldsymbol{\phi}_m^T + a_m(\nabla^2_{\mathbf{u}} d_m)\right] \\ &\approx \sigma_m^{-2}\left[-\left(1 - \frac{\kappa_m^2}{\sigma_m^2}\right)\boldsymbol{\phi}_m\boldsymbol{\phi}_m^T + a_m(\nabla^2_{\mathbf{u}} d_m)\right]\end{aligned} \tag{19}$$

Using (18), (19), and (13), $\mathbf{F}_m$ can be approximated by:

$$\mathbf{F}_m \approx \lambda'_m \boldsymbol{\phi}_m \boldsymbol{\phi}_m^T \tag{20}$$

where

$$\lambda'_m = \sigma_m^{-2}\left(1 - (\kappa_m/\sigma_m)^2\right). \tag{21}$$

From (9) and (20), we can see that $\lambda_m \approx \lambda'_m$. In addition, CRB can be approximated by:

$$\mathbf{C} \approx \left[\sum_{m=1}^{L} \lambda'_m \boldsymbol{\phi}_m \boldsymbol{\phi}_m^T + \sum_{m=L+1}^{M} \sigma_m^{-2} \boldsymbol{\phi}_m \boldsymbol{\phi}_m^T \right]^{-1} \tag{22}$$

Note that a better approximation of the CRB can be derived with a higher order Taylor-series expansion in the same way as given above, when the statistical moments of b_m higher than the 2nd order are known a priori. Here, we only use the first order Taylor-series expansion to derive an approximation when the 2nd order moment of b_m (i.e. κ_m) is known. The derived approximate CRB expression features a simple mathematical structure but reveals the influence of the bias distribution on the CRB clearly.

We will illustrate the effectiveness of (22) by numerical experiments using measured bias distributions in Section V. In fact, we can analytically illustrate the effectiveness of (22) when b_m is Gaussian distributed. Note that although in practice the Gaussian distribution is unlikely to be a good approximation of the true bias distribution, it lends the CRB to be easily simplified analytically so that the effectiveness of (22) can be examined. In this case, r_m can be equivalently modeled as $r_m = d_m + \overline{b}_m + \varepsilon'_m$ where ε'_m is Gaussian distributed with zero mean and variance $\sigma_m^2 + \kappa_m^2$. Therefore, $\lambda_m = (\sigma_m^2 + \kappa_m^2)^{-1}$ according to the earlier analysis for the case when the bias is deterministic in Section III. Since $\kappa_m \ll \sigma_m$, we can see that $\mathbf{C}$ can indeed be approximately computed with (22), due to the fact that

$$\begin{aligned} \lambda_m &= \sigma_m^{-2} \left(1 + \frac{\kappa_m^2}{\sigma_m^2} \right)^{-1} \\ &\approx \sigma_m^{-2} \left(1 - \frac{\kappa_m^2}{\sigma_m^2} \right) \\ &\approx \lambda'_m. \end{aligned} \tag{23}$$

V. MSE BOUND ANALYSIS

It is important to note that the MSE bound, namely the trace of $\mathbf{C}$ and denoted by $\mathrm{Tr}(\mathbf{C})$, reduces as λ_m increases. This is because $\sum_{m=1}^{M} \mathbf{F}_m$ increases in a positive definite sense, therefore both $\mathbf{C}$ and $\mathrm{Tr}(\mathbf{C})$ decrease, which means that the MSE bound improves, as λ_m increases.

When r_m $(m = 1, \cdots, L)$ are discarded, we can compute $\mathbf{C}$ by setting λ_m to 0. Since λ_m is no smaller than 0, using a bias-corrupted measurement results in the same or better MSE bound than discarding it.

An important property of λ_m is that $\lambda_m \leq \sigma_m^{-2}$. This can be justified by:

$$\begin{aligned} \lambda_m &= \sigma_m^{-4} \cdot \mathbb{E}_{r_m;\mathbf{u}} \left[\left(\mathbb{E}_{b_m|r_m;\mathbf{u}} \left[r_m - d_m - b_m \right] \right)^2 \right] \\ &\leq \sigma_m^{-4} \cdot \mathbb{E}_{r_m;\mathbf{u}} \left[\mathbb{E}_{b_m|r_m;\mathbf{u}} \left[(r_m - d_m - b_m)^2 \right] \right] \\ &= \sigma_m^{-4} \cdot \mathbb{E}_{b_m} \left[\mathbb{E}_{r_m|b_m;\mathbf{u}} \left[(r_m - d_m - b_m)^2 \right] \right] \\ &= \sigma_m^{-2} \end{aligned} \tag{24}$$

where $\mathbb{E}_{b_m}[\cdot]$ and $\mathbb{E}_{r_m|b_m;\mathbf{u}}[\cdot]$ stand for the expectation operator with respect to $p(b_m)$ and $p(r_m|b_m;\mathbf{u})$, respectively. Specifically, the second inequality is according to the Jensen's inequality, and the equality holds if b_m is deterministic (please refer to pages $77-78$ in [21]). This implies the presence of the random bias b_m degrades the MSE bound.

It is interesting to consider two special cases:

1) The first case is when b_m has a very large variance compared to σ_m (i.e., $\kappa_m \gg \sigma_m$) so that for any fixed r_m, $p(b_m)$ is approximately constant within the nonzero support of $p(r_m|b_m;\mathbf{u})$. This case corresponds to the scenario where the prior bias PDF is not informative, and we can find that $\lambda_m \approx 0$. If b_m is Gaussian distributed, this is the case when κ_m^2 is sufficiently large, and λ_m is indeed close to zero. This means that discarding a bias-corrupted measurement degrades the MSE bound slightly when the prior bias PDF is not informative.
2) The second case is when b_m has a very small variance compared to σ_m (i.e., $\kappa_m \ll \sigma_m$) so that for any fixed r_m, $p(r_m|b_m;\mathbf{u})$ varies slowly with respect to b_m within the nonzero support of $p(b_m)$. This case corresponds to the scenario where the prior bias PDF is very informative. Based on the analysis in Section IV, we can find that $\lambda_m \approx \lambda'_m = \sigma_m^{-2}\left(1 - (\kappa_m/\sigma_m)^2\right) \approx \sigma_m^{-2}$. This means that the presence of the random bias b_m results in a slight degradation to the MSE bound, compared to the case in which there are no measurement biases.

VI. Numerical experiments

For illustration purposes, we consider a wireless localization system with four beacons located respectively at $\mathbf{p}_1 = [0, 10]^T$, $\mathbf{p}_2 = [0, -10]^T$, $\mathbf{p}_3 = [-10, 0]^T$, $\mathbf{p}_4 = [10, 0]^T$, and a target at $\mathbf{u} = [-1, -5]^T$, as shown in Figure 1. We assume r_1 is biased while r_2, r_3, and r_4 are unbiased, and $\sigma_m = 1\,(m = 1, \cdots, 4)$. $p(b_1)$ is assumed to have the same shape as the measured bias distribution reported in [11]. Specifically,

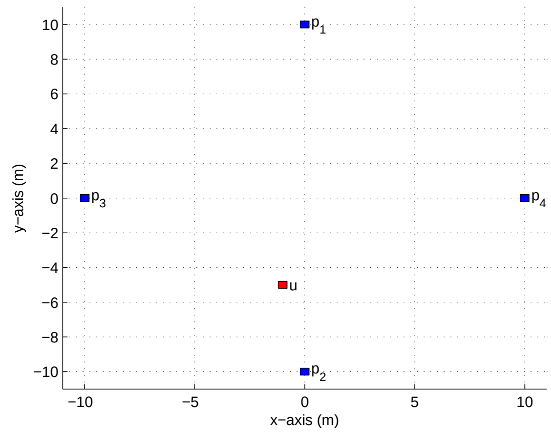

Fig. 1. The localization system with 4 beacons considered in the numerical experiments. Note that only r_1, namely the range measurement between $\mathbf{u}$ and $\mathbf{p}_1$, is biased.

TABLE I

THE VALUE OF P_i, $i = 0, \cdots, 8$.

P_0	P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8
0.12	0.03	0.31	0.12	0.24	0.12	0.03	0	0.03

$p(b_1)$ is expressed as

$$p(b_1) = \begin{cases} 0 & \text{if } b_1 \in (-\infty, \Omega_0] \cup (\Omega_9, \infty) \\ \frac{P_i}{\Delta} & \text{if } b_1 \in (\Omega_i, \Omega_{i+1}] \text{ with } i = 0, \cdots, 8, \end{cases} \tag{25}$$

where $\Omega_i = 0.1 + i\Delta$, and P_i is given in Table I for $i = 0, \cdots, 8$. When $\Delta = 0.1$, $p(b_1)$ is shown in Figure 2. It can be readily derived that $\overline{b}_1 = 0.1 + 3.49\Delta$ and $\kappa_1 = 1.83\Delta$, which means that κ_1 is a linear function of Δ. When Δ increases from 0.1 to 1, we have computed κ_1, and the results are shown in Figure 3. Note that all the above values related to a coordinate or length have the units of meters.

To examine the effectiveness of the approximate CRB expression (22), we have evaluated the CRB from (10) numerically, and the approximate CRB from (22) when Δ increases with $\kappa_1/\sigma_m < 1$ satisfied. The MSE bounds corresponding to those CRBs are shown with respect to κ_1/σ_m in Figure 4.a. We can see that when $\kappa_1/\sigma_m \leq 0.5$, the MSE bound corresponding to the approximate CRB is very close to the one computed from (10). This illustrates the effectiveness of (22) when $\kappa_1 \ll \sigma_m$.

To examine the analysis in Section V, we have also evaluated the CRB from (10) numerically when Δ increases with $\kappa_1/\sigma_m > 1$ satisfied, and the corresponding MSE bound is shown with respect to κ_1/σ_m

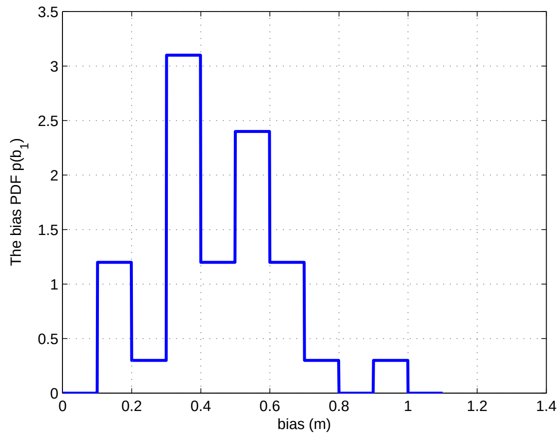

Fig. 2. The PDF of b_1, which is the bias of the range measurement between $\mathbf{u}$ and $\mathbf{p}_1$, when $\Delta = 0.1$ m.

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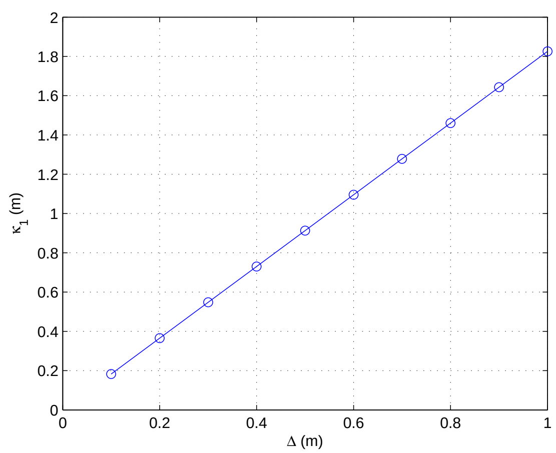
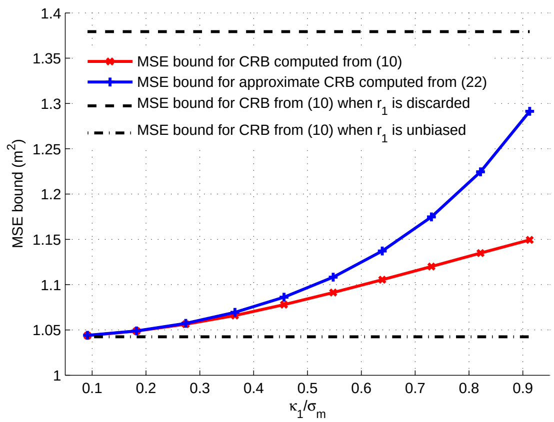


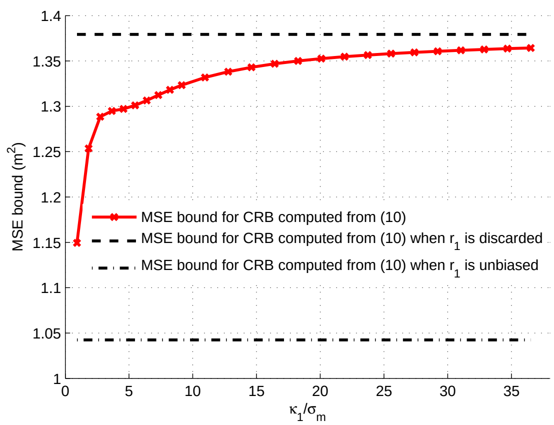
Fig. 3. κ_1, which is the standard deviation of the random bias b_1, when Δ increases from 0.1 to 1 m.

in Figure 4.b. We have also computed the CRBs when r_1 is discarded and when r_1 is not bias corrupted, respectively, and shown the corresponding MSEs in both Figure 4.a and Figure 4.b. It is clearly shown in Figure 4.a that as κ_1 reduces, the MSE bound approaches the one corresponding to the case when r_1 is not bias corrupted. On the other hand, as κ_1 increases, the MSE bound approaches the one corresponding to the case when r_1 is discarded, as shown in Figure 4.b. These observations corroborate the analysis in Section V.

Note that the derived CRB is an optimistic bound in scenarios when we do not have the a priori



(a) when $\kappa_1/\sigma_m < 1$



(b) when $\kappa_1/\sigma_m > 1$

Fig. 4. The MSE bounds corresponding to the CRBs computed when $\kappa_1/\sigma_m < 1$ and $\kappa_1/\sigma_m > 1$, respectively.

knowledge about which range measurements are biased. To illustrate this remark explicitly, we compare the MSE bound corresponding to the CRB computed from (10) with the MSE of two ML location estimators as κ_1/σ_m increases, since the ML estimator is widely used and able to achieve the corresponding CRB asymptotically. For the first ML estimator, the above mentioned a priori knowledge is not available, and a binary parameter s_m is introduced to indicate that r_m $(m = 1, \cdots, 4)$ is bias corrupted if $s_m = 0$. Specifically, the first ML estimator produces a joint estimate of $\mathbf{u}$ and $\{s_m, m = 1, \cdots, 4\}$ as the maximizer of the log-likelihood function

$$L(\mathbf{u}, s_1, s_2, s_3, s_4) = \sum_{m=1}^{4} L_m(\mathbf{u}, s_m), \tag{26}$$

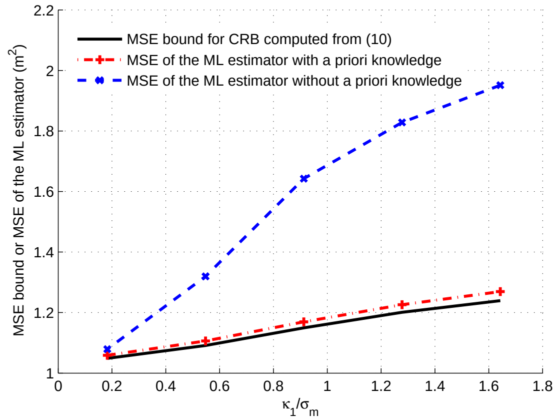


Fig. 5. The MSEs of the ML location estimators and the MSE bound corresponding to the CRB computed from (10). Note that the MSE bound is the same as the corresponding one depicted in Fig. 4.

where

$$L_m(\mathbf{u}, s_m) = \ln\left(\frac{1}{\sqrt{2\pi\sigma_m^2}}\right) + \ln\Bigg(s_m \exp\{-\frac{(r_m - d_m)^2}{2\sigma_m^2}\}$$
$$+(1 - s_m)\int_{-\infty}^{+\infty} p(b_m)\exp\{-\frac{(r_m - d_m - b_m)^2}{2\sigma_m^2}\}\mathrm{d}b_m\Bigg), \tag{27}$$

and $p(b_m) = p(b_1)$, $m = 2, 3, 4$. For the second ML estimator, the a priori knowledge is available, i.e., this estimator knows a priori that only r_1 is biased, and produces an estimate of $\mathbf{u}$ as the maximizer of $L(\mathbf{u}, s_1, s_2, s_3, s_4)$ when $s_1 = 0$ and $s_m = 1$, $m = 2, 3, 4$. For every Δ, the MSE of each estimator is computed by averaging the square errors of the location estimates for 1000 random realizations of range measurements, and the results are shown in Figure 5 with respect to κ_1/σ_m. We can see that the MSE of the second ML estimator is slightly above the MSE bound computed from (10), while the MSE of the first ML estimator is much greater than that MSE bound. These observations indicate that the derived CRB is indeed an optimistic lower bound for localization algorithms without the a priori knowledge about which range measurements are biased.

VII. Conclusion

We have derived a general expression for the CRB of wireless localization algorithms using range measurements subject to bias corruption. Specifically, the knowledge about which range measurements are biased, and the probability density functions (PDF) of the biases are assumed to be known a priori. For each range measurement, the error due to estimating the TOA of the detected signal is modeled

as a Gaussian distributed random variable with zero mean and known variance. We have also derived an approximate CRB expression when the bias PDF is very informative. Using these CRB expressions, we have studied the impact of the bias distribution on the MSE bound corresponding to the CRB. The analysis has been corroborated by numerical experiments.

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